Journal of Engineering Physics and Thermophysics, Vol. 73, No. 1, 2000

COMBINED METHOD FOR CALCULATING TEMPERATURE FIELDS IN THE STRUCTURE OF AIRCRAFT

V. M. Yudin

UDC 536.2:518.61

A method is developed which combines the advantages of economical difference-grid (a fast operation, high accuracy) and finite-element (arbitrariness of geometry) methods.

At the present time there are two trends in numerical methods for calculating the temperature fields in structures, i.e., classical difference methods in various modifications and finite-element methods (FEMs), which have intercomplementary advantages and drawbacks.

The difference methods ensure high accuracy of solution and fast operation in using schemes of splitting [1] and other economical algorithms, including the cases of calculating complex composite structures [2]. However, the field of their efficient use is limited by the structures composed of bodies of a canonical form.

The finite-element methods allow one to calculate the structures of an arbitrary form but their accuracy and fast operation are substantially lower and the possibilities of their improvement are not clear.

Therefore, it seems worthwhile to develop a combined computational method based on use of positive aspects of each of the above-mentioned groups of methods.

The most simple way of solving this problem is, by dividing the structure into subregions (substructures) of canonical and noncanonical forms, to use the difference solution methods for calculation of temperatures in the first ones and the finite-element methods, in the second ones, thus ensuring algorithms for noniterative matching of the solutions in the subregions into a common solution for the entire structure. The following conditions for technological implementation of the method must also be filfilled: the arbitrariness of a computational procedure for subregions, the mismatch of the grids and splittings into the elements of adjacent subregions, etc.

For the difference method of elementary balances (MEB), this problem was solved in [2].

A thin-walled structure is split by additional fins into subregions. In each subregion, one introduces its own grid of splitting, while on each fin, its own independent system of joining nodes, wherein the solutions in the adjacent subregions are matched or the boundary conditions are satisfied. Between the joining nodes and grid nodes of the subregion located on the fin an interpolation coupling is established, preferably not lower than the first order of smoothness. Under the conditions of the symmetric scheme of splitting by the MEB at each time step with the methods of parametric interpolation and trial run an equation of coupling between the boundary heat fluxes and temperatures at the joining nodes is constructed. The expressions of the heat fluxes obtained for each subregion in terms of the temperatures at the joining nodes are substituted into the balance equations at these nodes and in this manner a system of equations for the temperatures at all the joining nodes of the structure is formed. Upon determination, by solving this system, of the temperatures at the joining nodes the values of the heat fluxes at these nodes for each subregion are calculated. By means of interpolation, the heat fluxes at the boundary grid nodes are found, and by the symmetric scheme of splitting using the trial-run method the temperatures at all the grid nodes of the subregion are calculated.

In the FEM, the noniterative matching of the solutions between the subregions is conducted via the temperatures at the finite-element nodes and via the mean-integral heat flux through the element side, which corresponds to the heat flux at the midpoint of the side.

N. E. Zhukovskii Central Aerohydrodynamics Institute, Zhukovskii, Russia. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 73, No. 1, pp. 101-106, January-February, 2000. Original article submitted December 23, 1998.

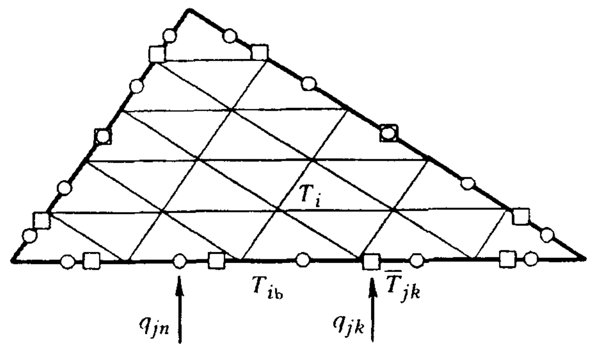

Fig. 1. Computational scheme of the subregion for the FEM: the centers of boundary sides of the finite elements are denoted by the circles; the joining nodes are denoted by the small squares.

To ensure the above-noted reproducibility conditions in the noniterative matching of the difference and finite-element solutions and also of the latter with each other in inconsistent splitting of the adjacent subregions, it is necessary within the framework of the finite-element solution to determine the relationship between the heat fluxes and temperatures at the joining nodes, which generally do not coincide with the nodes of the finite-element splitting of the subregion.

Let us consider a triangular subregion, on the fins of which length-variable heat fluxes are prescribed. We divide the subregion into the elements as is shown in Fig. 1. Here, the joining nodes are denoted by the small squares, and the centers of the element sides coincident with the boundary of the region, by the circles, where $i = 1, ..., I$ are the numbers of the finite-element splitting nodes.

We assume that the number of splitting elements of the subregion along the side, equal to N, provides a rather accurate simulation of the process of heat propagation in the subregion, while the number of joining nodes $K_j \leq N$, $j = 1, 2, 3$ is selected so that the interpolation polynomials, constructed at these nodes, describe with sufficient accuracy the distribution over the side of the temperature and the heat flux, which, in accordance with the physics of the process, are smooth functions.

Having denoted the temperature at the internal nodes of the finite-element model by T_i and at the boundary nodes by T_{ib}, and the heat fluxes at the midpoints of the boundary sides of the elements on the jth side of the contour by q_{jn} (n is the ordinal number of the point from the beginning of the side), we can write a system of finite-element equations for the subregion considered in the following form:

$$\begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{T}_b \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{C} \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{q} \end{pmatrix} + \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{pmatrix}, \tag{1}$$

where $\mathbf{q}$ is the heat flux density vector at the centers of the boundary sides of the elements; $\mathbf{T}$ and $\mathbf{T}_b$ are the temperature vectors at the internal and boundary nodes, respectively, $\mathbf{0}$ is the zero vector.

Eliminating the temperatures at the internal nodes from the system of equations (1), we obtain a system of $3N$ equations of coupling between the temperatures at the boundary nodal points of the finite-element model and the heat fluxes at the centers of the boundary sides of the elements

$$(\mathbf{B}_4 - \mathbf{B}_3 \mathbf{B}_1^{-1} \mathbf{B}_2)\, \mathbf{T}_b = \mathbf{C}\, \mathbf{q} + (\mathbf{F}_2 - \mathbf{B}_3 \mathbf{B}_1^{-1} \mathbf{F}_1)\,. \tag{2}$$

To obtain the equations of coupling between the heat fluxes and temperatures at the joining nodes, we make use of the procedures of parametric interpolation.

Since the heat flux distribution is continuous only within the limits of the subregion side (there are discontinuities over the contour at the corner points), the parametric representation of the vector components $\mathbf{q}$ in terms of the vector components of the fluxes at the joining nodes $\bar{\mathbf{q}}$ can be obtained independently for each side by the same method of smooth fulfillment that was used in matching the grid-difference solutions [2]. As a result, we have

$$\mathbf{q} = \mathbf{A}\,\bar{\mathbf{q}}, \tag{3}$$

where $\mathbf{A}$ is the matrix consisting of three nonzero blocks of dimensionality $N \times K_j$, (j = 1, 2, 3) positioned on the principal diagonal.

The temperature distribution over the contour is a continuous function having discontinuities at the corner points. Therefore, onto the interpolation functions on the sides we must superimpose the conditions of their conjugation at the corner points. In order to ensure this requirement, on each side we introduce into the number of interpolation nodes, in addition to the joining nodes, an initial point of the side and present the interpolation polynomials in the form of

$$T_j(\bar{l}) = a_j + \sum_{k=1}^{K_j} b_{jk}\,\bar{l}_j^{\,k}, \quad j = 1, 2, 3, \tag{4}$$

where $\bar{l} = l/L_j$ is the relative coordinate from the beginning of the side; a_j and b_{jk} are the unknown coefficients of the interpolation functions.

Assuming that the temperatures at the joining nodes T_{jn} are prescribed, for determining the unknown coefficients we have the ordinary conditions for coincidence of the values of the interpolation function (of the K_jth power polynomial) with the prescribed values of the temperatures at the interpolation nodes, which are the joining nodes:

$$a_j + \sum_{k=1}^{K_j} b_{jk}\,\bar{l}_{jn}^{\,k} = \bar{T}_{jn}, \quad n = 1, \dots, K_j, \quad j = 1, 2, 3, \tag{5}$$

and the additional conditions for conjugation of the interpolation functions at the corner points of the subregion contour:

$$a_1 + \sum_{k=1}^{K_1} b_{1k} = a_2, \quad a_2 + \sum_{k=1}^{K_2} b_{2k} = a_3, \quad a_3 + \sum_{k=1}^{K_3} b_{3k} = a_1. \tag{6}$$

An immediate solution of the system of equations (5)-(6) will determine the values of the sought coefficients a_j and b_{jk}

$$\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \mathbf{a} \end{pmatrix} = \mathbf{M}^{-1} \begin{pmatrix} \bar{\mathbf{T}}_1 \\ \bar{\mathbf{T}}_2 \\ \bar{\mathbf{T}}_3 \\ \mathbf{0} \end{pmatrix}, \tag{7}$$

where $\mathbf{a}$, $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ are the vectors of the coefficients; $\bar{\mathbf{T}}_1$, $\bar{\mathbf{T}}_2$, and $\bar{\mathbf{T}}_3$ are the temperature vectors at the nodal points of the sides; $\mathbf{M}^{-1}$ is the inverse matrix of the system of equations (5)-(6).

However, in implementation of the method on a computer, it is more profitable to use the following algorithm. Transposing in Eqs. (5) the coefficients a_j into the right side and solving for each side its own system of equations, separated from system (5), we obtain the expression for the vector $\mathbf{b}_j$ in terms of $\bar{\mathbf{T}}_j$ and a_j:

$$\mathbf{b}_j = \mathbf{M}_j^{-1}\,\bar{\mathbf{T}}_j - a_j S_j, \quad j = 1, 2, 3, \tag{8}$$

where $S_j = \sum_{i=1}^{K_j} \sum_{k=1}^{K_j} m_{ik}^{-1}$ is the sum of the elements m_{ik}^{-1} of the inverse matrix $\mathbf{M}_j^{-1}$ of the separated system of equations for the jth side. The inverse matrices $\mathbf{M}_j^{-1}$ depend only on the location of the joining nodes on the side.

Substituting Eq. (8) into Eq. (6), we come to the system of three equations

$$\begin{pmatrix} -s_1 & 1 & 0 \\ 0 & -s_2 & 1 \\ 1 & 0 & -s_3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}, \tag{9}$$

here $s_j^{-1} = 1 - S_j$; $d_j = \sum_{k=1}^{K_j} \bar{T}_{jk} \sum_{i=1}^{K_j} m_{ik}^{-1}$.

From Eqs. (9) we have

$$a_j = \sum_{i=1}^{3} s_{ji}^{-1} d_i, \tag{10}$$

where s_{ji}^{-1} are the elements of the inverse matrix in system (9).

Thus, the coefficients a_j and, by virtue of Eq. (8), also b_{jk} are expressed in terms of the temperatures at the joining nodes of the subregion.

Using the expressions for the coefficients, obtained by any of the methods considered, and interpolational functions (4), we find the expression for the temperatures at the boundary nodes of the finite-element model of the subregion in terms of the temperatures at the joining nodes

$$\mathbf{T}_b = \mathbf{D}\bar{\mathbf{T}}, \tag{11}$$

where $\mathbf{D}$ is the matrix of coupling; $\bar{\mathbf{T}}$ is the temperature vector at the joining nodes.

Substitution of $\mathbf{T}_b$ from Eq. (11) and $\mathbf{q}$ from Eq. (3) into Eqs. (2) gives the system of equations of coupling between the heat fluxes and temperatures at the joining nodes

$$\bar{\mathbf{C}}\bar{\mathbf{q}} = \overline{\mathbf{D}\mathbf{T}} + \bar{\mathbf{F}} \tag{12}$$

($\bar{\mathbf{C}} = \mathbf{CA}$; $\bar{\mathbf{D}} = (\mathbf{B}_4 - \mathbf{B}_3\mathbf{B}_1^{-1}\mathbf{B}_2)\mathbf{D}$; $\bar{\mathbf{F}} = \mathbf{F}_2 - \mathbf{B}_3\mathbf{B}_1^{-1}\mathbf{F}_1$).

Solving Eq. (12) for $\bar{\mathbf{q}}$:

$$\bar{\mathbf{q}} = \bar{\mathbf{C}}^{-1}\overline{\mathbf{D}\mathbf{T}} + \bar{\mathbf{C}}^{-1}\bar{\mathbf{F}}, \tag{13}$$

we obtain the sought expressions for the heat fluxes at the joining nodes in terms of the temperatures at the same nodes and also in the case of solving the problem in the subregion by the FEM.

Substituting the expressions for the heat fluxes, found for all the subregions by the MEB or FEM and also from the boundary conditions on the free sides of the subregions, into the heat balance equations at the corresponding joining nodes, we come to the system of equations for the temperatures at these nodes

$$\sum_{r=1}^{R} \bar{q}_{kr}\delta_r = 0, \quad k = 1, \ldots, K, \tag{14}$$

where R is the number of the subregions joining at the given node, including also the boundary conditions; δ is the subregion thickness; K is the general number of the joining nodes in the structure.

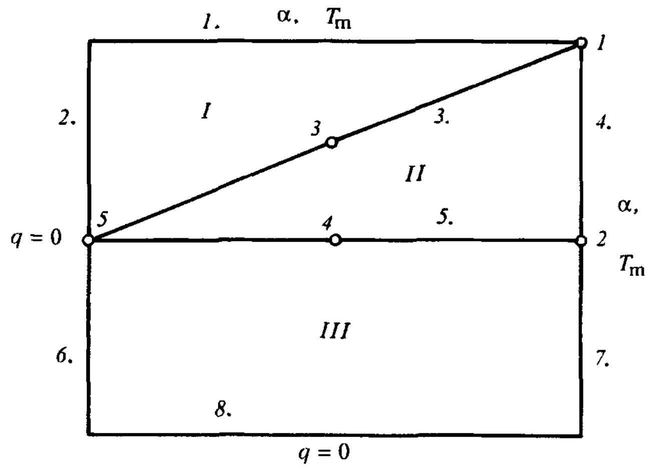

Fig. 2. Test problem: *I-III*) numbers of the subregions; 1-8) (with the points) numbers of the fins; 1-5) points at which the solutions of the problem, obtained by various methods, are compared.

Having determined from the solution of the system of equations (14) the values of the temperatures at the joining nodes, it is possible to calculate the temperatures in any subregion on the corresponding difference-grid or finite-element grid.

Without describing here this procedure for the elementary-balance method (see ref. [2]), let us recall its basic stages. From the values of the temperatures at the joining nodes, the heat fluxes at these nodes are found and by their interpolation the heat fluxes at the boundary nodes of the difference grid are determined, and then by means of the symmetric scheme of the splitting method using the trial run the values of the temperatures at the internal nodes are calculated.

In the case of the FEM we determine the values of the heat fluxes at the joining nodes from relations (13), and the heat fluxes on the boundary sides of the finite elements, from dependences (3). From the solution of the system of equations (2) the values for the temperatures at the boundary nodes of the finite-element model of the subregion are found

$$\mathbf{T}_b = (\mathbf{B}_4 - \mathbf{B}_3\,\mathbf{B}_1^{-1}\,\mathbf{B}_2)^{-1}\,\mathbf{C}\mathbf{q} + (\mathbf{B}_4 - \mathbf{B}_3\,\mathbf{B}_1^{-1}\,\mathbf{B}_2)^{-1}\,(\mathbf{F}_2 - \mathbf{B}_2\,\mathbf{B}_1^{-1}\,\mathbf{F}_1) \tag{15}$$

and then the temperatures at the internal nodes are calculated

$$\mathbf{T} = \mathbf{B}_1^{-1}\,\mathbf{F}_1 - \mathbf{B}_1^{-1}\,\mathbf{B}_2\,\mathbf{T}_b\,. \tag{16}$$

Thus, the temperature distribution in the structure at this step is completely determined and it is possible to pass to the following time step.

The method developed was checked by comparing with the analytical method and finite-element solutions of the problem for a square plate of dimensions $L \times L$ made of material with constant thermophysical characteristics. At the boundaries of the plate $x = L$ and $y = L$ heat transfer at constant values of the heat transfer coefficient α and the medium temperature T_m occurs, while at the boundaries $x = 0$ and $y = 0$ the thermal insulation condition $q = 0$ is prescribed. The initial temperature of the plate is constant and equal to T_0.

An analytical solution of this problem has the following form:

$$\frac{T - T_0}{T_m - T_0} = 1 - 4\sum_{p_1}\sum_{p_2} \frac{h^2 \cos p_1 x \cos p_2 y}{[L^2 (p_1^2 + h^2) + h]\,[L^2 (p_2^2 + h^2) + h]}\,\frac{\exp[-a\,(p_1^2 + p_2^2)\,t]}{\cos p_1 L \cos p_2 L}\,,$$

TABLE 1. Comparison of the Results of Solving the Problem (Fig. 2) by the Combined Method with the Analytical Solution and Solution by the FEM

Number of point	Method of solution	Time, sec				
		5	10	50	100	200
1	Analytical	376.7	406.0	520.3	615.7	767.1
	Combined	370.5	403.6	519.0	614.7	766.4
	FEM	370.6	403.6	519.1	614.9	766.5
2	Analytical	340.7	361.3	469.2	571.4	734.2
	Combined	337.7	359.8	468.2	570.7	733.7
	FEM	337.0	359.4	468.0	570.5	733.5
3	Analytical	311.8	329.3	437.2	543.7	713,7
	Combined	310.9	328.0	436.1	542.8	713.0
	FEM	311.8	327.7	436.0	542.7	712.9
4	Analytical	303.1	314.0	414.3	523.7	698.9
	Combined	302.9	313.1	413.5	523.0	698.4
	FEM	302.8	312.8	413.2	522.7	698.2
5	Analytical	301.6	307.6	396,1	507.5	686.9
	Combined	301.5	307.3	395.4	506.9	686.4
	FEM	301.4	307.1	395.1	506.5	686.1

where $h = \alpha/\lambda$; $a = \lambda/c\rho$; $p_1 = p_2$ are the roots of the equation $\tan pL = h/p$.

In solving the problem numerically by the method developed, the plate was divided into three subregions: two triangular subregions and one rectangular subregion. In Fig. 2 we give the numbering of the subregions, vertices, and fins; the circles indicate the points at which we compare the results of calculations of the temperature by the indicated methods. The triangular subregions were split into finite elements in such a manner that the number of elements along the side was equal to 9 for the first one and to 8 for the second one, while the rectangular subregion was split into 11 elements along theOx axis and into 7 elements along the Oy axis. The number of joining nodes was taken to be equal to 6 on the first fin, 5 on the second fin, 7 on the third fin, 5 on the fourth, 6 on the fifth, 4 on the sixth, 4 on the seventh, and 6 on the eighth fin. The first and last joining nodes on the fin were located at the centers of the first and last elements having the smallest dimension among all of the nodes joining on this fin, whereas the internal nodes were located at the equal distance from each other. To calculate the temperatures in the triangular subregions, use was made of the FEM, while in the rectangular subregion, the MEB.

In solving the problem by means of the FEM, we split the plate into 8 elements along the Ox axis and into 16 elements along the Oy axis. Here we observe the coincidence of the splitting nodes in the zone of the second subregion with nodes in calculating the temperatures in the plate by the combined method.

The following values of the parameters and characteristics are adopted: $L = 0.025$ m; $p = 8000$ kg/m^3; $c = 500$ J/(kg·K); $\lambda = 20$ W/(m·K); $\alpha = 160$ W/(m^2·K); $T_m = 1200$ K; $T_0 = 300$ K, $\Delta t = 5$ sec.

In Table 1, we give the calculated values for the temperatures at the points denoted in Fig. 2.

The presented results confirm the efficiency of the combined method and indicate that the method suggested for matching the solutions on inconsistent grids does not worsen the accuracy of the finite-element solutions and even ensures their refinement provided that the difference-grid solution method is used in the adjacent subregion.

NOTATION

T, temperature; t, time; l, coordinate; L, side length; ρ, density; c, heat capacity; λ, thermal conductivity coefficient; α, heat-transfer coefficient; q, heat flux density; $\mathbf{F}_1$ and $\mathbf{F}_2$, vectors of free terms; $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{B}_4$, submatrices of the global matrix in a finite-element system of linear equations; $\mathbf{C}$, matrix of the coefficients of boundary conditions. Subscripts: b, boundary; 0, initial value; m, medium.

REFERENCES

1. A. A. Samarskii, *Theory of Difference Schemes* [in Russian], Moscow (1977).
2. V. M. Yudin and V. F. Kravchenko, *Insh.-Fiz. Zh.*, **56**, No. 3, 478-483 (1989).